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Thermal convection over flat plates possessing an irregular leading edge

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Abstract

Boundary layer similarity reductions for thermal convection adjacent to inclined heated plates with an irregular leading edge are presented. The theory is outlined using three examples: Newtonian convection over an isothermal plate, Darcian convection over a nonuniformly heated plate, and mixed convection of Newtonian shear flow over a nonuniformly heated plate. These reductions represent a natural extension of results obtained by the author for Blasius flow over a flat plate with an irregular leading edge.

1. Introduction

The purpose of this paper is to show how one may construct asymptotic similarity reductions for thermal convection over steeply inclined plates with irregular leading edges. Both natural and mixed convection with variable heat transfer are included. The results presented here represent a direct extension to thermally active wall-bounded flows of the work reported by Weidman [1] for Blasius boundary layer flow over a plate with an irregular leading edge.

The mathematical analysis for convection flows that develop over plates with irregular leading edges will be elucidated with three examples, beginning with natural convection of a Newtonian fluid above an inclined isothermal plate. In the second example, natural convection flow in a fluid-saturated porous medium bounded by an inclined, nonuniformly heated plate is considered. The last example deals with mixed convection driven by the uniform shear of a Newtonian fluid streaming past an inclined plate with a specific power-law temperature distribution. Velocities (u, v, w) along coordinate directions (x, y, z) are employed, with x the streamwise coordinate antiparalled the projection of gravity g along the plate, y the plate-normal coordinate, and z the spanwise coordinate. All plates are *flat* with an irregular leading edge located at $x = x_e(z)$ with planform as sketched in Fig. 1; plates with an irregular leading edge that are *curved* in the streamwise direction will be considered in the discussion of results.

2. Natural convection of a Newtonian fluid over an inclined isothermal plate

We consider the boundary layer form of momentum and energy equations for incompressible flow in the Boussinesq limit. Scaling coordinates with the plate length l, velocities with ν/l, and introducing the temperature variable $\theta = (T - T_\infty)/\Delta T$, the thermal boundary layer equations for steady flow, with zero streamwise pressure gradient, may be written in the nondimensional form

$$u_x + v_y = 0, \qquad (1)$$

$$uu_x + vu_y = u_{yy} + (Gr \sin\epsilon)\theta, \qquad (2)$$

$$u\theta_x + v\theta_y = \frac{1}{\sigma}\theta_{yy}. \qquad (3)$$

The three dimensionless parameters are the plate inclination angle ϵ measured from horizontal, the Grashof number $Gr = g\beta\Delta T l^3/\nu^2$, and the Prandtl number $\sigma = \nu/\kappa$. The dimensional constants are the thermal contrast $\Delta T = T_e - T_\infty > 0$ between the uniform leading

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Nomenclature

C	integration constant
$F(x)$	integration function
f	velocity similarity variable
Gr	Grashof number
g	gravity; temperature similarity variable
K	permeability of porous matrix
l	plate length
Ra	Rayleigh number
T	temperature
ΔT	temperature contrast
U	free-stream velocity
u, v	streamwise and plate-normal velocities
x, y, z	streamwise, plate normal and spanwise coordinates

Greek symbols

β	volume expansion coefficient
γ	shear rate
$\delta(x,z)$	boundary layer similarity function
ϵ	plate inclination angle
$\eta(x,y,z)$	independent similarity variable
θ	dimensionless temperature variable
κ	thermal diffusivity of Newtonian fluid
κ^*	thermal diffusivity of liquid-saturated porous matrix
λ	wall temperature distribution exponent
μ	scaling factor ($Ra \sin\epsilon$ or $Gr \sin\epsilon$)
v	kinematic viscosity
σ	Prandtl number
$\phi(x,z)$	velocity similarity function
$\psi(x,z)$	temperature similarity function

Subscripts

e	leading edge
∞	free-stream

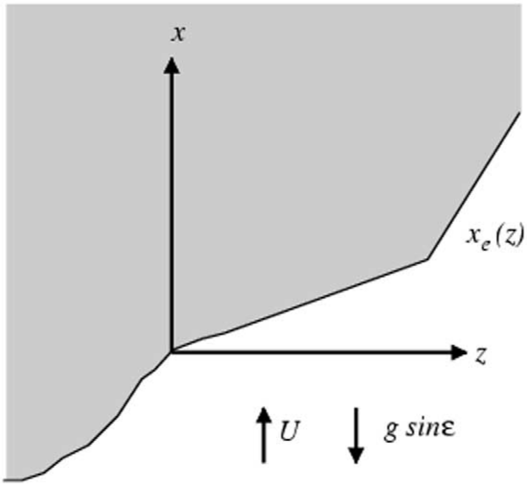

Fig. 1. Coordinates (x, z) attached to a flat plate, inclined at angle ϵ to the horizontal, with an irregular leading edge $x_e(z)$; not shown in this planform view is the plate-normal y-coordinate. The surface component of gravity $g \sin\epsilon$ and the direction of a possible external streaming motion U along the plate are also indicated.

edge temperature $T_e = T(x_e(z), 0, z)$ and the uniform ambient fluid temperature T_∞, the uniform gravitational constant g; the fluid properties of interest are its volume expansion coefficient β, its kinematic viscosity v, and its thermal diffusivity κ. Since plate-normal buoyancy effects are absent, it is tacitly assumed here and in the following examples that the plate is steeply inclined. The exclusion of horizontal and slightly inclined plates has consequences for finding irregular leading edge solutions; further consideration of this point is deferred to the discussion of results.

Scaling out $Gr \sin\epsilon \equiv \mu$ in the usual manner [2]

$$x \to x, \quad y \to \mu^{-1/4} y, \quad u \to \mu^{1/2} u, \quad v \to \mu^{1/4} v, \tag{4}$$

gives the one-parameter family of governing partial differential equations

$$u_x + v_y = 0, \tag{5}$$

$$uu_x + vu_y = u_{yy} + \theta, \tag{6}$$

$$u\theta_x + v\theta_y = \frac{1}{\sigma}\theta_{yy}. \tag{7}$$

For a plate held at constant temperature, we posit the three-dimensional similarity solution form

$$u(x,y,z) = \phi(x,z) f'(\eta), \quad \theta(x,y,z) = g(\eta),$$
$$\eta(x,y,z) = \frac{y}{\delta(x,z)},$$

which transforms the boundary layer equations into the one-parameter family of PDEs

$$f''' + \delta(\delta\phi)_x f f'' - \delta^2 \phi_x f'^2 + \frac{\delta^2}{\phi} g = 0, \tag{8}$$

$$g'' + \sigma\delta(\delta\phi)_x f g' = 0, \tag{9}$$

in which a prime denotes differentiation with respect to similarity variable η. ODEs are obtained when

$$\delta(\delta\phi)_x = C_1, \quad \delta^2\phi_x = C_2, \quad \frac{\delta^2}{\phi} = C_3,$$

where C_1, C_2, and C_3 are constants. These are, in fact, PDEs for δ and ϕ in terms of x and z. Eliminating δ

from the first two equations gives, for appropriate choice of constants C_1 and C_2, the result $\phi\phi_x = 1/2$ which, when integrated on x, yields

$$\phi(x,z) = [x + F(z)]^{1/2}.$$

Fixing $\phi = 0$ at the leading edge $x = x_e(z)$ determines the unknown function $F(z) = -x_e(z)$, and hence

$$\phi(x,z) = [x - x_e(z)]^{1/2}.$$

The choice $C_3 = 1$ in the third restriction furnishes the boundary layer thickness distribution over the surface of the plate, viz.

$$\delta(x,z) = [x - x_e(z)]^{1/4}.$$

The remaining constants are then $C_1 = 3/4$ and $C_2 = 1/2$ in which case the coupled pair of nonlinear ODEs governing the natural convection flow are

$$f''' + \frac{3}{4}ff'' - \frac{1}{2}f'^2 + g = 0, \tag{10}$$

$$g'' + \frac{3}{4}\sigma f g' = 0. \tag{11}$$

It is well known that these equations possess solutions satisfying the physically relevant boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 0,$$

$$g(0) = 1, \quad g(\infty) = 0.$$

All boundary conditions at $\eta = 0$ are applied on the plate where $x \geqslant x_e(z)$. It is understood, however, that these asymptotic results admit a constant shift in the x-coordinate since the boundary layer equations break down in the vicinity of the leading edge where velocity curvature becomes important. Eqs. (10) and (11) were originally found by Pohlhausen [3] for planar flow over a flat plate with a straight leading edge; in the present problem they govern the three-dimensional velocity and temperature fields emanating from an irregular leading edge.

3. Natural convection of a Darcian fluid over a non-uniformly heated inclined plate

Consider now the motion of a liquid fully saturating a uniformly porous medium for which the Darcian equations reported by Wooding [4] are employed. Scaling spatial coordinates with l, percolation velocities with κ^*/l, and using the dimensionless temperature θ introduced in the previous section gives the Darcy boundary layer equations

$$u_x + v_y = 0, \tag{12}$$

$$u = (Ra \sin\epsilon)\theta, \tag{13}$$

$$u\theta_x + v\theta_y = \theta_{yy}, \tag{14}$$

where $Ra = gK\Delta T l/\nu\kappa^*$ is the porous media Rayleigh number. The new dimensional quantities K and κ^* are, respectively, the permeability of the medium and the thermal diffusivity of the liquid-saturated porous matrix. Again it has been tacitly assumed that the plate is of sufficient inclination for the plate-normal buoyancy term to be neglected at leading order. The factor $Ra \sin\epsilon \equiv \mu$ may be scaled out in the usual manner [2]

$$x \to x, \quad y \to \mu^{-1/2}y, \quad u \to \mu u, \quad v \to \mu^{1/2}v,$$

giving in the new dimensionless variables $u = \theta$, and elimination of θ in the remaining equations yields the parameter-free system of PDEs

$$u_x + v_y = 0, \tag{15}$$

$$uu_x + vu_y = u_{yy}. \tag{16}$$

These are recognized as the Prandtl [5] boundary layer equations for isothermal flow of a viscous fluid over a flat plate that were analyzed in detail by Blasius [6]; here, however, there is slip at the wall in the Darcy approximation.

Consider now the situation where the temperature of the plate bounding the porous medium takes on a power-law profile $[x - x_e(z)]^\lambda$ emanating from the irregular leading edge. In this case the *ansatz* for similarity taken as

$$u(x,y,z) = \theta(x,y,z) = [x - x_e(z)]^\lambda f'(\eta), \quad \eta(x,y,z) = \frac{y}{\delta(x,z)}$$

transforms Eqs. (15) and (16) into

$$f''' + \lambda\delta^2[x - x_e(z)]^{\lambda-1}(ff'' - f'^2) + \delta\delta_x[x - x_e(z)]^\lambda ff'' = 0. \tag{17}$$

This becomes an ODE when

$$\delta^2[x - x_e(z)]^{\lambda-1} = C_1, \quad \delta\delta_x[x - x_e(z)]^\lambda = C_2,$$

in which the constants C_1 and C_2 must be chosen so that a self-consistent pair of PDEs for the boundary layer thickness distribution $\delta(x,z)$ is obtained. Selecting $C_1 = 1$ and integrating the first equation on x with the stipulation that the boundary layer originates at the irregular leading edge gives

$$\delta(x,z) = [x - x_e(z)]^{(1-\lambda)/2}$$

from which the second constant $C_2 = (1 - \lambda)/2$ is found. Hence the governing similarity equation is

$$f''' + \frac{(1+\lambda)}{2}ff'' - \lambda f'^2 = 0 \tag{18}$$

to be solved with boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0.$$

The second boundary condition corresponds to a heated plate when u is taken as the temperature, and to wall slip when u is taken as the streamwise velocity. The range $0 < \lambda < 1$ for physically relevant wall temperature distributions are the same as those reported by Cheng and Minkowycz [7] who first solved the problem for a straight leading edge: $\lambda = 0$ corresponds to an isothermal wall; $\lambda = 1/3$ gives a uniform wall heat flux; and $\lambda = 1$ is the limiting case for which the boundary layer thickness is constant.

4. Uniform shear flow of a Newtonian fluid over a heated inclined plate

As a final illustration we consider mixed convection above a steeply inclined plate subjected to a streamwise uniform shear flow $U = \gamma y$ of strain rate γ. The author is not aware of any previous discussion of this particular mixed convection problem in the heat transfer literature, though a general discussion of *mixed* convection flows is given in [8] and *forced* convection with uniform shear is a problem that has received considerable attention; see for example, [9]. Normalizing coordinates with $\sqrt{\nu/\gamma}$ and velocities with $\sqrt{\nu\gamma}$ for incompressible boundary layer flow in the Boussinesq approximation again yields Eqs. (1)–(3) but with the modified Grashof number $Gr = g\beta\Delta T/\nu^{1/2}\gamma^{3/2}$. Using this Grashof number, the affine transformation (4) again produces Eqs. (5)–(7), but now the dimensionless far field boundary condition on the streamwise velocity is

$$u(x,y,z) \to y, \quad \text{as } y \to \infty. \tag{19}$$

A similarity reduction posited in the form

$$u(x,y,z) = \phi(x,z)f'(\eta), \quad \theta(x,y,z) = \psi(x,z)g(\eta),$$
$$\eta(x,y,z) = \frac{y}{\delta(x,z)}$$

transforms Eqs. (5)–(7) into the coupled system of PDEs

$$f''' + \delta(\delta\phi)_x ff'' - \delta^2\phi_x f'^2 + \frac{\psi\delta^2}{\phi}g = 0, \tag{20}$$

$$g'' + \sigma\left[\delta(\delta\phi)_x fg' - \frac{\phi\psi_x\delta^2}{\psi}f'g\right] = 0 \tag{21}$$

that reduce to ODEs under the restriction that the four distinct coefficient functions of (x,z) are constant. Moreover, boundary condition (19) is of similarity form only for the choice $\phi = \delta$ and hence the restrictions for similarity are

$$\delta(\delta^2)_x = C_1, \quad \delta^2\delta_x = C_2, \quad \psi\delta = C_3, \quad \frac{\psi_x}{\psi}\delta^3 = C_4.$$

Integrating the second relation on x for the convenient choice $C_2 = 1/3$ gives

$$\delta^3 = x + F(z),$$

which satisfies the leading edge condition $\delta(x_e(z), z) = 0$ if $F(z) = -x_e(z)$. This result, with the choice $C_3 = 1$, yields

$$\phi(x,z) = \delta(x,z) = [x - x_e(z)]^{1/3},$$
$$\psi(x,z) = [x - x_e(z)]^{-1/3}.$$

The remaining constants are then $C_1 = 2/3$ and $C_4 = -1/3$ in which case the coupled ODEs governing the mixed convection flow are

$$f''' + \frac{2}{3}ff'' - \frac{1}{3}f'^2 + g = 0, \tag{22}$$

$$g'' + \sigma\left[\frac{2}{3}fg' + \frac{1}{3}f'g\right] = 0 \tag{23}$$

to be solved with boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad g(0) = 1,$$
$$f(\eta) \to \eta, \quad g(\eta) \to 0 \quad \text{as } \eta \to \infty.$$

No analytical solutions have been found for specific choices of σ, so it appears that the entire family of Prandtl number-dependent solutions must be obtained by direct numerical integration.

In concluding this section, it may be noted that wall mass transfer can be included in the problem if the wall transpiration $v(x,0,z)$ is proportional to $[x - x_e(z)]^{-1/3}$; in this case, however, a streamwise pressure gradient of the same x-dependence is required to maintain the flow.

5. Discussion and conclusion

The intuitive result that a boundary layer flow generated by an external rectilinear stream parallel to a flat plate with an irregular leading edge will develop identical asymptotic velocity profiles at each station $x - x_e(z)$ from the leading edge was proven rigorously by Weidman [1]. In this paper we have shown that the same result holds for the velocity and temperature fields in natural and mixed convection flow along inclined heated plates. For *natural* convection the component of buoyancy that drives the motion is rectilinear over the surface of the plate so the flow develops only in that direction. For *mixed* convection, it is necessary that the external stream be aligned with the buoyancy force to ensure that the fluid moves rectilinearly along the plate. It should also be evident that these same types of similarity reductions are possible for *forced* convection flows emanating from an irregular leading edge, since then temperature is a passive scalar.

We now return to the point mentioned earlier that all examples discussed in this paper deal with steeply inclined flat plates for which the O(1) buoyancy force acts tangent to the surface of the plate. Upward-facing heated horizontal plates like that originally discussed by Stewartson [10] are necessarily excluded because the fluid motion is driven not by a component of buoyancy acting along the plate, but rather by a horizontal thermally-induced pressure gradient. At the leading edge, this pressure gradient is perpendicular to the leading edge. For example, the convection flow that develops above a heated upward-facing horizontal plate of wedge-shape planform will consist of two streams, each respectively moving normal to their straight leading edges; the two boundary layer flows will collide obliquely along the wedge centerline and most likely erupt in a thermal plume. Convection downstream of a very irregular leading edge will have even more complicated behavior and it is difficult to see how the asymptotic flow can assume similarity form. The same holds true of a slightly-inclined upward-facing heated plate, a situation first considered by Jones [11] in the absence of wall transpiration and later by Weidman and Amberg [2] to include the effects wall transpiration distributions that admit self-similar solutions; in each case the plate-tangent component of buoyancy and the thermally-induced pressure gradient are of equal importance.

An extension to flows with other rectilinear forces acting along the surface of the plate is plausible. If all the dominant forces driving the fluid motion are collinear and parallel to the surface of a flat plate, developing in a manner that depends only on the distance $x - x_e(z)$ from the leading edge, then one should expect three-dimensional similarity reductions for the asymptotic flow far downstream of the leading edge.

We conclude this discussion by considering the problem of natural convection above a steeply inclined curved plate with an irregular leading edge where the curvilinear streamwise x-coordinate is everywhere antiparallel the component of gravity projected on the plate. A family of shapes for which self-similar solutions exist in two-dimensional planar flow over uniformly heated curved plates with straight leading edges has been reported by Braun et al. [12]. When the straight leading edge becomes irregular, it is apparent that no asymptotic self-similar solution exists because a requirement for similarity is that the streamwise forces acting on the fluid particles at the leading edge be *uniform*. For an isothermal curved plate, for example, the leading edge buoyancy force proportional to $g\Delta T \sin \epsilon_e$ is nonuniform since the surface inclination angle ϵ_e at the leading edge varies with $x_e(z)$ which itself is nonuniform in every case except that for which the leading edge is straight and perpendicular to the ensuing direction of flow.

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